Chenyang Wang† and John Preskill‡
Institute for Quantum Information, California Institute of Technology, Pasadena, CA 91125, USA

We study the $\pm J$ random-plaquette Z_2 gauge model (RPGM) in three spatial dimensions, a three-dimensional analog of the two-dimensional $\pm J$ random-bond Ising model (RBIM). The model is a pure Z_2 gauge theory in which randomly chosen plaquettes (occuring with concentration p) have couplings with the "wrong sign" so that magnetic flux is energetically favored on these plaquettes. Excitations of the model are one-dimensional "flux tubes" that terminate at "magnetic monopoles" located inside lattice cubes that contain an odd number of wrong-sign plaquettes. Electric confinement can be driven by thermal fluctuations of the flux tubes, by the quenched background of magnetic monopoles, or by a combination of the two. Like the RBIM, the RPGM has enhanced symmetry along a "Nishimori" line in the p-T plane (where T is the temperature). The critical concentration p_c of wrong-sign plaquettes at the confinement-Higgs phase transition along the Nishimori line can be identified with the accuracy threshold for robust storage of quantum information using topological error-correcting codes: if qubit phase errors, qubit bit-flip errors, and errors in the measurement of local check operators all occur at rates below p_c, then encoded quantum information can be protected perfectly from damage in the limit of a large code block. Through Monte-Carlo simulations, we measure p_{c0}, the critical concentration along the $T = 0$ axis (a lower bound on p_c), finding $p_{c0} = .0293 \pm .0003$. We also measure the critical concentration of antiferromagnetic bonds in the two-dimensional RBIM on the $T = 0$ axis, finding $p_{c0} = .1030 \pm .0002$. Our value of p_{c0} is incompatible with the value of $p_c = .1093 \pm .0002$ found in earlier numerical studies of the RBIM, in disagreement with the conjecture that the phase boundary of the RBIM is vertical (parallel to the T axis) below the Nishimori line. The model can be generalized to a rank-r antisymmetric tensor field in d dimensions, in the presence of quenched disorder.

I. INTRODUCTION

Spin systems with quenched randomness have been extensively studied, leading to valuable insights that apply to (for example) spin glass materials, quantum Hall systems, associative memory, error-correcting codes, and combinatorial optimization problems [1–3]. Gauge systems with quenched randomness, which have received comparatively little attention, will be studied in this paper.

The gauge models we consider are intrinsically interesting because they provide another class of simple systems with disorder-driven phase transitions. But our investigation of these models has a more specific motivation connected to the theory of quantum error correction.

In practice, coherent quantum states rapidly decohere due to uncontrollable interactions with the environment. But in principle, if the quantum information is cleverly encoded [6,7], it can be stabilized and preserved using fault-tolerant recovery protocols [8]. Kitaev [4,5] proposed a particularly promising class of quantum error-correcting codes (*surface codes*) in which the quantum processing required for error recovery involves only *local* interactions among qubits arranged in a two-dimensional block, and the protected information is associated with global topological properties of the quantum state of the block. If the error rate is small, then the topological properties of the code block are well protected, and error recovery succeeds with a probability that rapidly approaches one in the limit of a large code block. But if the error rate is above a critical value, the *accuracy threshold*, then quantum error correction is ineffective.

In [9], a precise connection was established between the accuracy threshold achievable with surface codes and the confinement-Higgs transition in a three-dimensional Z_2 lattice gauge model with quenched randomness. The model has two parameters: the temperature T and the concentration p of "wrong-sign" plaquettes. On wrong-sign plaquettes (which are analogous to antiferromagnetic bonds in a spin system) it is energetically favorable for the Z_2 magnetic flux to be nontrivial. In the mapping between quantum error recovery and the gauge model, the quenched fluctuations correspond to the actual errors introduced by the environment; these impose sites of frustration, *magnetic monopoles*, corresponding to an "error syndrome" that can be measured by executing a suitable quantum circuit. Thermally fluctuating magnetic flux tubes, which terminate at magnetic monopoles,

---

*CALT-68-2374
†chenyang@caltech.edu (after 1 September 2002: chenyang@stanford.edu)
‡preskill@theory.caltech.edu

correspond to the ensemble of possible error patterns that could generate a particular error syndrome. (The temperature T is tied to the strength p of the quenched fluctuations through a *Nishimori relation* [10].) When the disorder is weak and the temperature low (corresponding to a small error rate), the system is in a magnetically ordered Higgs phase. In the surface code, magnetic order means that all likely error patterns that might have produced the observed error syndrome are topologically equivalent, so that the topologically encoded information resists damage. But at a critical value p_c of the disorder strength (and a temperature determined by Nishimori's relation), magnetic flux tubes condense and the system enters the magnetically disordered confinement phase. In the surface code, magnetic disorder means that the error syndrome cannot point to likely error patterns belonging to a unique topological class; therefore topologically encoded information is vulnerable to damage.

Although the code block is two dimensional, the gauge model is three dimensional because one dimension represents *time*. Time enters the analysis of recovery because measurements of the error syndrome might themselves be faulty; therefore measurements must be repeated on many successive time slices if they are to provide reliable information about the errors that have afflicted the code block. If qubit phase errors, qubit bit-flip errors, and errors in the measurement of local check operators all occur at rates below p_c, then encoded quantum information can be protected perfectly from damage in the limit of a large code block. As we consider more and more reliable measurements of the syndrome, the corresponding three-dimensional gauge model becomes more and more anisotropic, reducing in the limit of perfect measurements to the two-dimensional random-bond Ising model.

The numerical value p_c of the accuracy threshold is of considerable interest, since it characterizes how reliably quantum hardware must perform in order for a quantum memory to be robust. In the three-dimensional Z_2 gauge model, p_c is the value of the wrong-sign plaquette concentration where the confinement-Higgs boundary crosses the Nishimori line in the p-T plane. A lower bound on p_c is provided by the critical concentration p_{c0} on the $T=0$ axis. In [9], an analytic argument established that $p_{c0} \geq .0114$. In this paper we report on a numerical calculation that finds $p_{c0} = .0293 \pm .0003$.

In the case where the error syndrome can be measured flawlessly, the critical error rate is given by the critical antiferromagnetic bond concentration on the Nishimori line of the two-dimensional random-bond Ising model (RBIM). Numerical calculations performed earlier by other authors [11,12] have established $p_c = .1093 \pm .0002$. According to a conjecture of Nishimori [13] and Kitatani [14], this value of p_c should agree with the critical bond concentration p_{c0} of the 2D RBIM on the $T=0$ axis. The same reasoning that motivates this conjecture for the RBIM indicates that $p_c = p_{c0}$ for the 3D random-plaquette gauge model (RPGM) as well. However, we have calculated p_{c0} in the 2D RBIM numerically, finding $p_{c0} = .1030 \pm .0002$. Our value of p_{c0} agrees with an earlier numerical calculation by Kawashima and Rieger [23], but disagrees with the conjecture that $p_c = p_{c0}$.

In Sec. II we describe in more detail the properties of the 2D RBIM and the 3D RPGM, emphasizing the importance of the Nishimori line and the inferences that can be made about the behavior of order parameters on this line. Section III reviews the connection between the models and error recovery using surface codes. Our numerical results for p_{c0} and for the critical exponent ν_0 at the $T=0$ critical point are presented in Sec. IV. Section V summarizes our conclusions.

II. MODELS

A. Random-bond Ising model

The two-dimensional $\pm J$ random-bond Ising model (RBIM) has a much studied multicritical point at which both the temperature and the strength of quenched disorder are nonzero. This model is an Ising spin system on a square lattice, with a variable $S_i = \pm 1$ residing at each lattice site i. Its Hamiltonian is

$$H = -J \sum_{\langle ij \rangle} \tau_{ij} S_i S_j \ , \tag{1}$$

where J is the strength of the coupling between neighboring spins, and $\tau_{ij} = \pm 1$ is a quenched random variable. (That is, τ_{ij} depends on what *sample* of the system is selected from a certain ensemble, but is not subject to thermal fluctuations.) The τ_{ij}'s are independently and identically distributed, with the antiferromagnetic choice $\tau_{ij} = -1$ (favoring that neighboring spins antialign) occuring with probability p, and the ferromagnetic choice τ_{ij}=+1 (favoring that neighboring spins align) occuring with probability $1-p$. We refer to p as the concentration of antiferromagnetic bonds, or simply the bond concentration.

The free energy F of the model at inverse temperature β, averaged over samples, is

$$[\beta F(K,\tau)]_{K_p} = -\sum_{\tau} P(K_p,\tau) \ln Z(K,\tau) \tag{2}$$

where

$$Z(K,\tau) = \sum_{S} \exp\left(K \sum_{\langle ij \rangle} \tau_{ij} S_i S_j \right) \tag{3}$$

is the partition function for sample τ (with $K = \beta J$), and

$$P(K_p,\tau) = (2\cosh K_p)^{-N_B} \times \exp\left(K_p \sum_{\langle ij \rangle} \tau_{ij} \right) \tag{4}$$